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On the reflexion of wave characteristics from rough surfaces

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The energy of internal waves tends to be propagated along certain characteristic paths inclined at fixed angles to the vertical direction; the angle of inclination depending only on the wave frequency and the density stratification (not on the wavelength). The reflexion of such waves by smooth plane surfaces has been discussed recently by Sandstrom (1966).

In the present paper the role of surface roughness is examined. Surprisingly, it appears that quite small-scale irregularities can completely alter the reflecting properties of a surface; the tangential scale of the roughness elements may be much smaller than the wavelength of the incident or reflected waves. All scales of roughness are relevant, down to those comparable in magnitude to the thickness of the oscillatory boundary layer. For tidal waves in the ocean this thickness is of the order of 1 m.

The behaviour of the coefficient of transmission as a function of the angle of incidence appears at first sight to be extremely complicated. Some simple examples of periodic surface roughnesses are discussed and elucidated: a saw-tooth, a square-topped wave and a simple sine-wave. The transmission coefficient T for a sine-wave, for example, is shown in figure 9. An approximate expression for T is also derived in the case of a slowly modulated sine-wave (figure 10). These results are for a non-viscous fluid. The effects of viscosity are also considered qualitatively.

1. Introduction

The laws of reflexion of acoustical and electromagnetic waves from rough surfaces are fairly well understood. In other kinds of media the propagation and reflexion of radiation may be similar to that of light provided the differential equation governing the motion is elliptic in the space variables. However, for many other types of wave motion, particularly for internal waves in the ocean, the governing differential equation is hyperbolic (see, for example, Eckart 1960). Consequently, the wave energy in such a fluid tends to be concentrated along certain characteristic paths in the fluid, inclined at fixed angles to the vertical.

In internal waves of small amplitude, for example, if the rotation and compressibility are both neglected, the governing differential equation for the vertical displacement ζ can be written

$$\frac{\partial^2}{\partial t^2}\nabla^2\zeta + N^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\zeta = 0, \tag{1.1}$$

where N is the Brunt–Väisälä frequency,

$$N^2 = -\frac{g}{\rho}\frac{d\rho}{dz}, \tag{1.2}$$

(assumed constant) and x, y, z are rectangular co-ordinates, z vertically upwards. If the radian frequency σ is less than N, periodic solutions exist in the form

$$\zeta = e^{-i\sigma t} \times F(x\cos\theta + z\sin\theta), \tag{1.3}$$

where F is an arbitrary twice-differentiable function, provided only that

$$\cos\theta = \pm\sigma/N. \tag{1.4}$$

In other words, the motions tend to be propagated along paths making a fixed angle $\theta = \cos^{-1}\sigma/N$ with the vertical direction, regardless of wavelength.

The above conclusions can easily be generalized to a rotating stratified fluid (Eckart 1960; Sandstrom 1966). Thus Sandstrom shows that the differential equation for small-amplitude wave motion is hyperbolic whenever the frequency σ satisfies

$$\frac{(N^2-\sigma^2)f_V^2}{N^2-\sigma^2+f_H^2} < \sigma^2 < N^2, \tag{1.5}$$

where f_V and f_H are the vertical and horizontal components of the Coriolis parameter. The wave energy is then propagated along straight paths whose angle to the vertical depends on the azimuth. If f_H^2 is neglected in comparison with N^2, then the above condition reduces simply to

$$f_V^2 < \sigma^2 < N^2, \tag{1.6}$$

as found by Eckart (1960). The inclination of the ray paths is then independent of the azimuth.

The importance of this type of propagation for internal tides in the ocean was emphasized by Sandstrom (1966), who carried out some simple and elegant experiments demonstrating the validity of the theory for baroclinic waves in a laboratory wave tank. Partial barriers placed in the path of a wave were found to produce shadow zones bounded by one of the characteristic surfaces,† according to expectations.

Fofonoff (1967) has noted that the slope of the wave characteristics for tidal waves is almost equal to that of the continental slope in the north-west Atlantic. If internal tides are assumed to be generated mainly on the continental shelf, then one may consequently expect to find in the deep ocean certain zones where the tidal amplitude is intensified, others where it is weak or zero. Nevertheless, field observations designed to detect such effects have not yet positively confirmed these expectations.

Now the purpose of this note is to point out the peculiar effects which quite small irregularities in a reflecting surface may have upon the nature of the reflected waves. These will affect not only the proportion of energy reflected or transmitted, but also the amount dissipated at the surface itself.

† See also some recent experiments by Mowbray & Rarity (1967) on the generation of internal waves by moving bodies.

Moreover, in the more familiar type of reflexion the important parameter is the ratio of the scale of the surface roughness relative to the wavelength. But in the reflexion of internal waves the important ratio is the scale of the roughness relative to the thickness of the oscillatory boundary layer—a much smaller quantity. Hence, quite small-scale irregularities can have a pronounced effect on the coefficients of reflexion and transmission.†

For simplicity we consider any medium such that the rays have a constant slope α, that is to say they make a constant angle

$$\theta = \cot^{-1}\alpha \tag{1.7}$$

with the vertical. We have seen that this is exactly true of waves in stratified fluid when the rotation is ignored, in which case $\cos\theta = \sigma/N$. (It is also true in a rotating but perfectly homogeneous fluid, in which case the angle θ between the rays and the axis of rotation is given by $\sin\theta = \sigma/2\Omega$, where Ω denotes the radian frequency of the rotation.) We shall especially have in mind baroclinic tides in the ocean, for which (1.7) is approximately valid. The results may easily be extended to the general case, when θ is a function of the azimuth.

In §§2–7 we shall neglect all viscous effects. The discussion is then purely geometrical. The subject may be said to be a peculiar kind of geometrical optics, where the reflected and incident rays make equal angles, not with the normal to the reflecting surface, but with an axis fixed in space.

The necessary modifications resulting from viscosity are discussed in §8. Finally, in §9 follows a short discussion of some of the implications for the reflexion of internal waves in the ocean.

2. Reflexion from a plane surface

Consider first the situation shown in figure 1, where an incident ray OP of slope α meets a plane surface with slope β (Sandstrom 1966). The reflected ray must have the opposite slope $-\alpha$. If $|\beta| < \alpha$, as in figures 1 (*a*) and (*c*), then the reflected energy is carried onwards along PQ. If $\beta > \alpha$, as in figure 1 (*b*), then the energy is reflected back along PQ'.

The situations 1 (*a*) and 1 (*c*) we shall describe as 'onwards reflexion' or 'transmission', and 1 (*b*) we shall describe as 'backwards reflexion' or simply 'reflexion'.

We may note that an incident beam continued between two parallel rays will be reflected as a narrower beam if $\beta > 0$ as in figures 1 (*a*) and (*b*) and as a broader beam if $\beta < 0$ as in figure 1 (*c*).

The direction of the rays is of course the same as the direction of the *group*-velocity. In an isotropic medium this generally differs from the direction of the

† This difference in the reflecting properties of a rough surface towards waves governed by elliptic and hyperbolic equations respectively may be traced to the fact that, with an equation which is elliptic in the space variables, the small-scale irregularities may be patched up by solutions that are sinusoidal in the direction tangential to the mean surface and are exponentially decreasing along the inwards normal. With a hyperbolic equation, on the other hand, such solutions are not available; wave solutions varying harmonically in one direction vary harmonically also in the direction at right angles.

phase-velocity, which is orthogonal to the wave crests. In fact, in the baroclinic waves considered here, the group velocity and phase-velocity are mutually perpendicular; the rays are parallel to the wave crests. Thus a narrower beam for the reflected waves implies also a shorter wavelength.

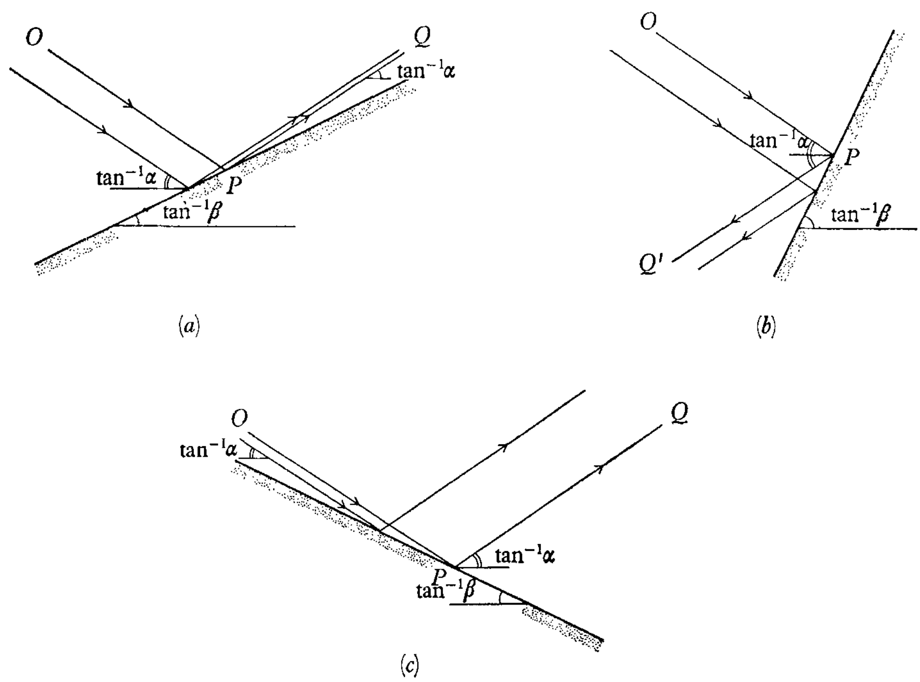


FIGURE 1. Examples of the reflexion of characteristic rays (with slope α) from a plane surface (with slope β). (a) $\alpha > \beta$; (b) $\alpha < \beta$; (c) $\alpha > -\beta > 0$.

If we define the transmission coefficient T as the ratio of the total transmitted (or onward-reflected) energy to the total incident energy, then clearly we have

$$T = \begin{cases} 1 & (\beta/\alpha < 1), \\ 0 & (\beta/\alpha > 1). \end{cases} \tag{2.1}$$

3. Reflexion from a saw-tooth

Suppose first that the roughness has the form of a symmetrical saw-tooth, with slopes β, as in figure 2.

If the slope β of the roughnesses is less than α, as in figure 2 (*a*), then the incident rays have unimpeded access to both the forward and rear slopes of the saw tooth, and so also have the transmitted rays. Those rays falling on the forward slopes are diminished in intensity on reflexion, and those falling on the rear slopes are increased in intensity. But all the energy is reflected forward so that on the whole $T = 1$.

If, however, β is greater than α (as in figure 2 (*b*)) then no rays fall on the forwards slope of the saw-tooth, and all those rays falling on the rear slopes are reflected backwards at the same angle α. The same rays are reflected again forwards but downwards, so that seemingly the energy ends up in the lower corners of the saw-tooth, where it is absorbed by viscosity (see §8).† We have then $T = 0$.

The graph of T versus the ratio β/α is then as shown in figure 3.

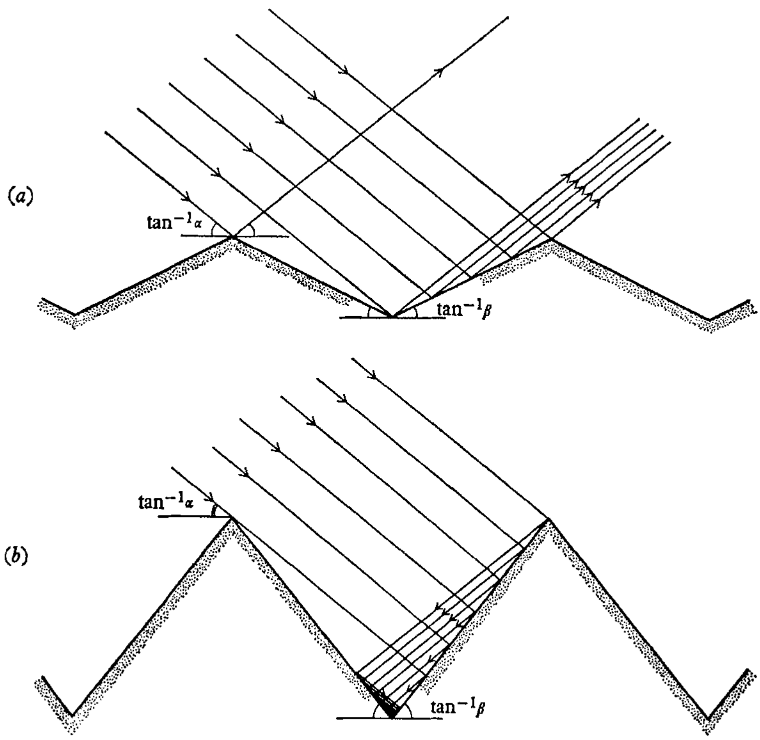


FIGURE 2. The reflexion of rays from a simple saw-tooth. (a) $\alpha > \beta$, (b) $\alpha < \beta$.

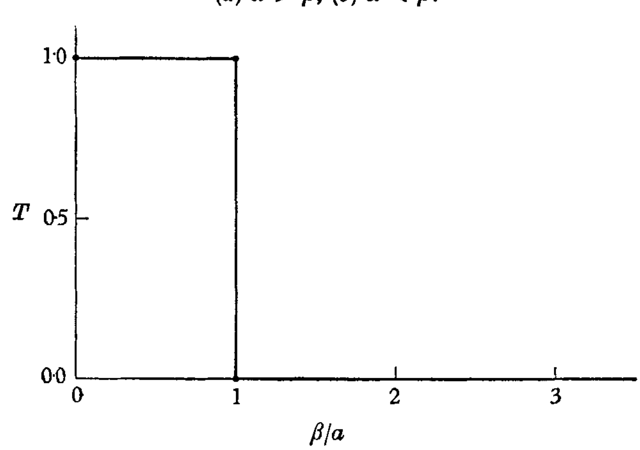


FIGURE 3. The transmission coefficient T for the saw-tooth roughness of figure 2, as a function of β/α.

† It is assumed that the wave amplitude is small enough that the absorption takes place before non-linear effects cause any reflexion.

4. Reflexion from a square-topped wave

Very different is the transmission coefficient from a square-topped wave-form such as that shown in figure 4.

Let α denote the width of the horizontal upper part of the wave-form, b the width of the lower part, where $a+b = 1$. Further, let h denote the height of the vertical sides. Clearly, all rays falling on the upper part are transmitted; hence we have always

$$T \geqslant a. \tag{4.1}$$

Consider now the rays falling in the remaining segments of the surface. If $h/\alpha \ll 1$, then most of these rays strike the lower horizontal part of the wave-form and are reflected onwards. But a small fraction of the beam, after striking the flat lower portion, hits the side wall and is reflected back. This fraction of the incident rays is equal to PD in figure 4 (a), where PE is the reflected ray through the corner E. An exactly equal proportion of rays hits the side wall first and is reflected back off the bottom. Since PD equals h/α (where α is the slope of the incident rays) the proportion of reflected radiation equals $2h/\alpha$. Hence the transmission coefficient is given by

$$T = 1-2h/\alpha \quad (0 \leqslant h/\alpha \leqslant \tfrac{1}{2}b). \tag{4.2}$$

The upper bound for h/α in the last equation is determined by the fact that, when $PD = \frac{1}{2}b$, then P lies at the mid-point M of CD, and $T = 1-b = a$ (see figure 4 (b)). At somewhat greater values of h/α a portion of the reflected beam equal to $2PM$ is reflected again off the other vertical wall and is transmitted onwards as in figure 4 (c). Since $PM = (h/\alpha-\frac{1}{2}b)$ it follows that $T = a+2(h/\alpha-\frac{1}{2}b)$, that is to say

$$T = 1+2(h/\alpha-b) \quad (\tfrac{1}{2}b \leqslant h/\alpha \leqslant b). \tag{4.3}$$

When $h/\alpha = b$, as in figure 4 (d), the width of the backwards-reflected beam is reduced to zero and so $T = 1$. When h/α somewhat exceeds b (figure 4 (e)), part of the beam is reflected again and hence

$$T = 1-2(h/\alpha-b) \quad (b \leqslant h/\alpha \leqslant \tfrac{3}{2}b). \tag{4.4}$$

At $h/\alpha = \frac{3}{2}b$, we have $T = a$ again (figure 4 (f)). Then part of the beam is reflected yet again and T increases linearly towards unity; and so on.

In general it can be seen that, at the critical slopes α given by

$$h/\alpha = (n+\tfrac{1}{2})b \quad (n = 0, 1, 2, 3, \ldots), \tag{4.5}$$

when the incident ray through B (after n reflexions) passes through the mid-point of CD, all the rays incident between B and E are reflected backwards; hence $T = a$. On the other hand, when

$$h/\alpha = nb \quad (n = 0, 1, 2, 3, \ldots), \tag{4.6}$$

so that the ray through B (after n reflexions) passes through one or the other of the end-points of CD, all the rays incident between B and E are eventually reflected onwards; hence $T = 1$.

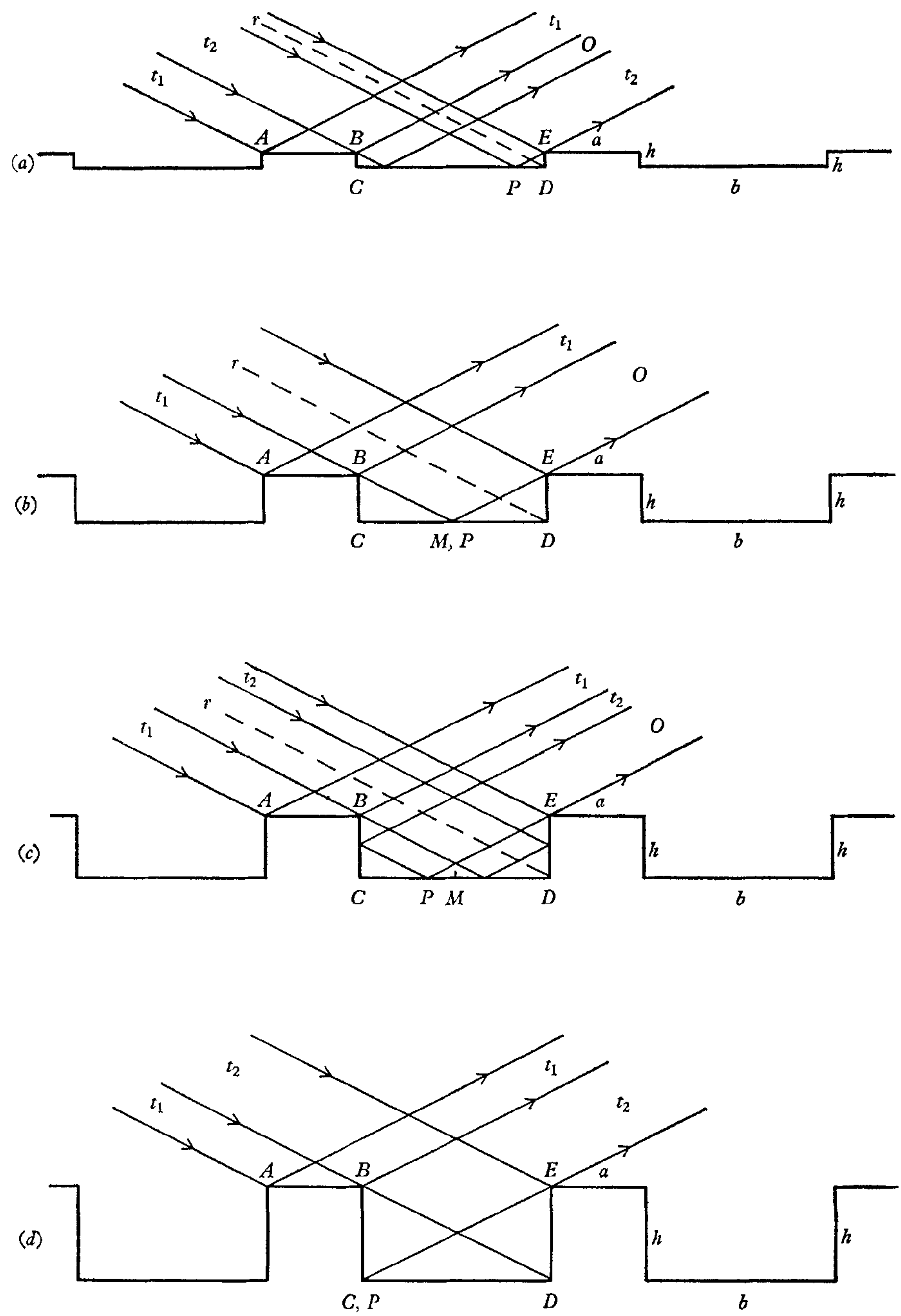
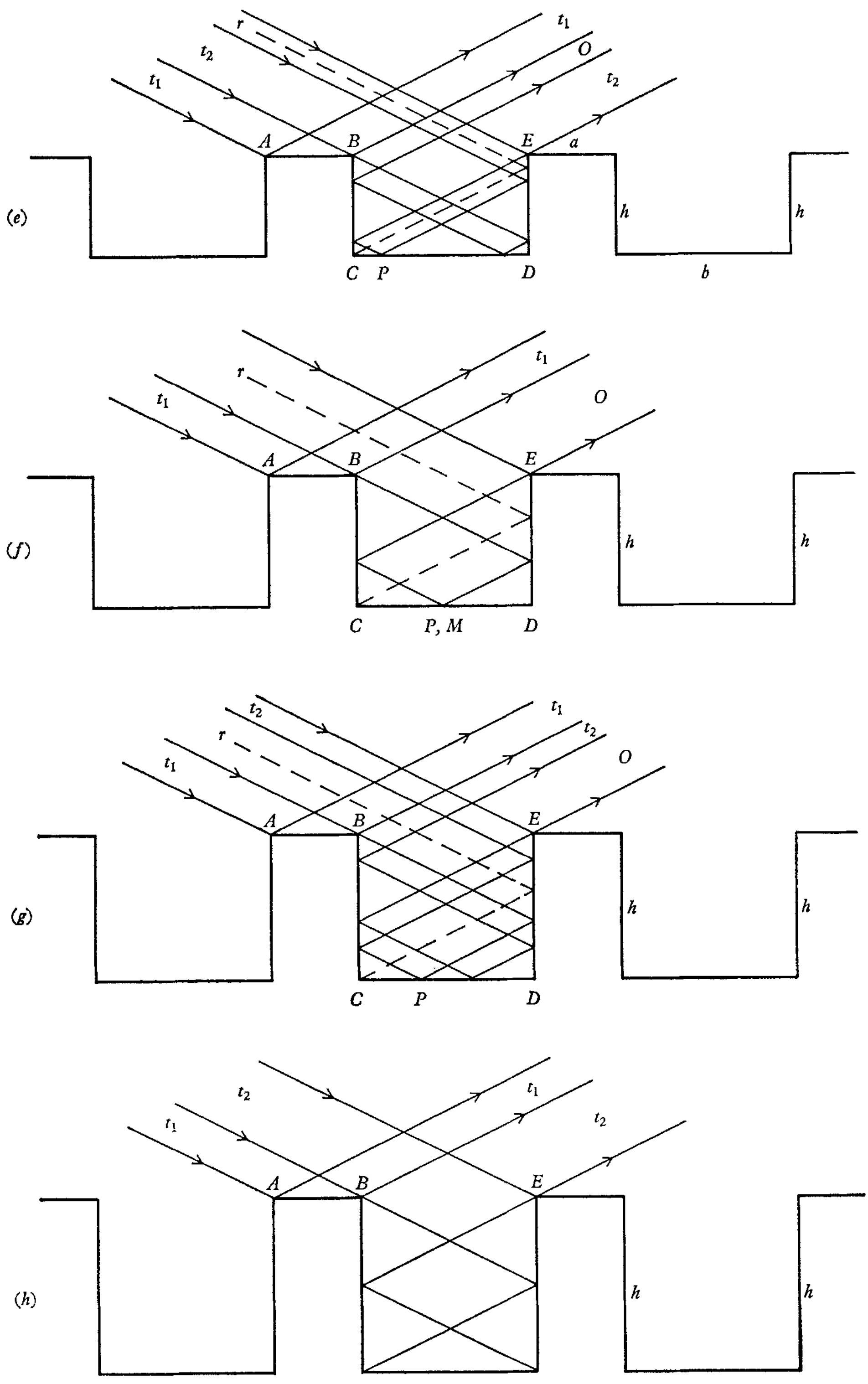


FIGURE 4. The reflexion of rays from a square-topped roughness of height h and width b for increasing ratios $h/b\alpha$. Transmitted rays are denoted by t_1, t_2 and reflected rays by r. Shadow zones are denoted by O. Figures 4 (b) and (d) correspond to $h/b\alpha = \frac{1}{2}$ and 1. Figures 4 (f) and (h) correspond to $h/b\alpha = \frac{3}{2}$ and 2.



FIGURES 4*e*–*h*. For legend see previous page.

Between the critical slopes α given by (4.5) and (4.6), the position of the point P on CD depends linearly on h/α (PD being a constant $\pm h/\alpha$). Hence, the graph of T versus h/α consists of a series of straight-line segments passing between the levels $T = a$ and $T = 1$, as in figure 5.

In the limiting case $a = 0$, the roughness consists of a sequence of thin vertical barriers of height h spaced at equal intervals b. The transmission coefficient T then oscillates linearly between the values 0 and 1.

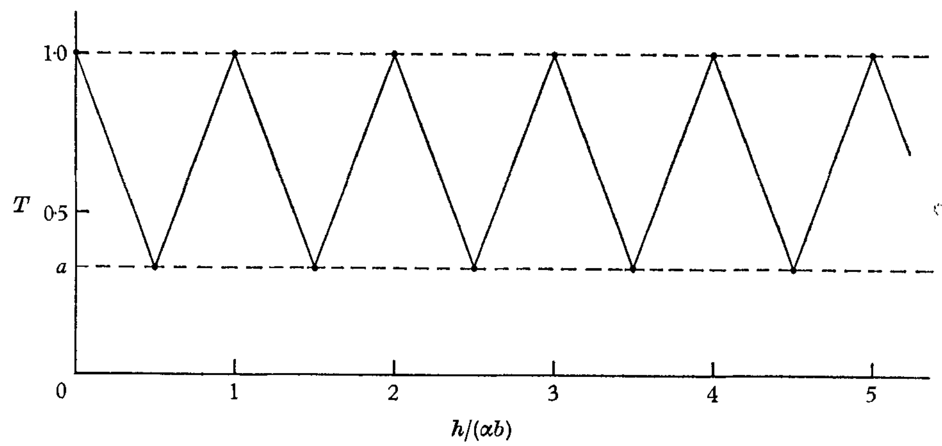


FIGURE 5. The transmission coefficient T for the square-topped roughness of figure 4.

5. Reflexion from a blunt saw-tooth

The apparently quite different behaviour of the transmission coefficient for the wave-forms indicated in §§3 and 4 may be reconciled by considering the wave-form shown in figure 6 (a 'blunt saw-tooth'). This is derived from the 'saw-tooth' of figure 2 by truncating each angle so as to leave horizontal reflecting surfaces of width $\frac{1}{2}c$ at both top and bottom of the wave-form.

This affects the transmission coefficient in two obvious ways. First, the upper horizontal surfaces reflect onwards a proportion $\frac{1}{2}c$ of the incident rays, so that at all events $T \geqslant \frac{1}{2}c$. Secondly, the lower surfaces prevent energy being lost in the lower angles, so long as c is large compared with the thickness δ of the boundary layer.

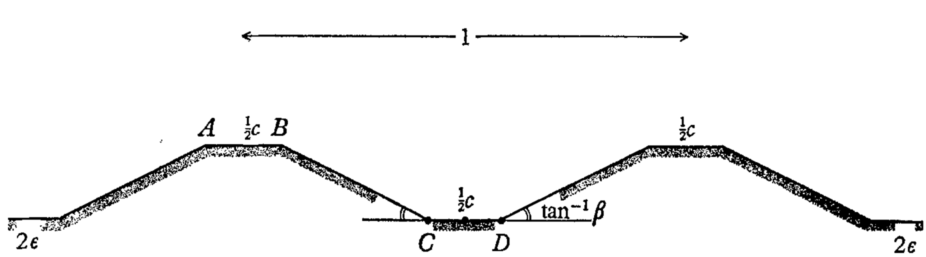


FIGURE 6. A blunt saw-tooth: definition diagram.

Evidently, as $c \to 0$, the wave form tends to the saw-tooth form of figure 2. As $c \to 1$, however, the form reduces almost to the square-topped wave-form of figure 4, provided $b = \frac{1}{2}$ and the height h of the wave is given by

$$h = \beta(\tfrac{1}{2} - c). \tag{5.1}$$

The difference between the two wave-forms is that the vertical sides of the square-topped wave-form are replaced by sloping sides over a small distance $(\frac{1}{2} - c)$. It is clear that so long as $\beta > \alpha$ this will make little difference to the transmission coefficient.

For any given value of c and α, the transmission coefficient may easily be determined by graphical construction. When $\beta/\alpha < 1$ all rays are obviously reflected onwards, and so $T = 1$. When $\beta/\alpha > 1$ the behaviour of T is qualitatively

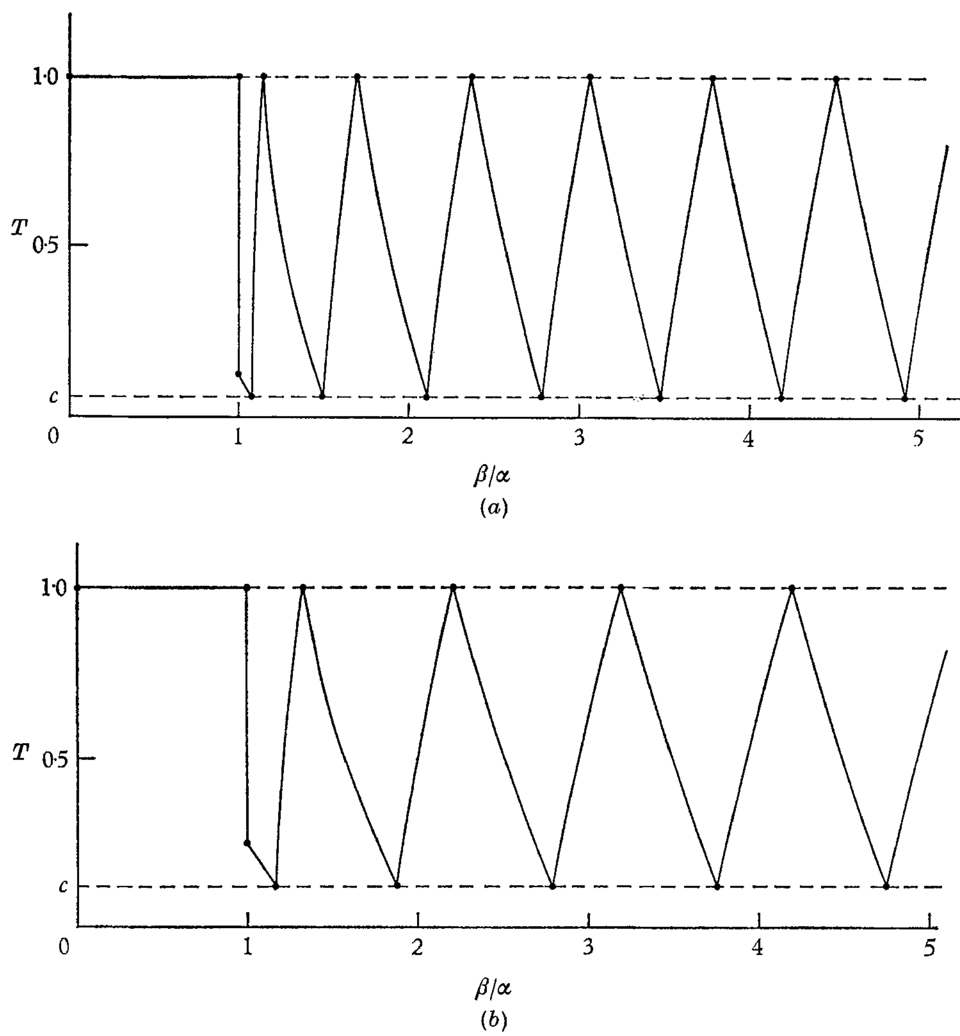
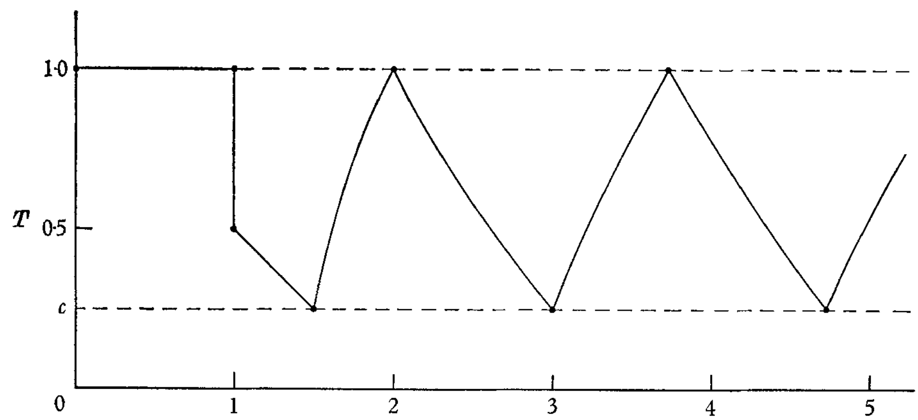


Figure 7 (*a*), (*b*). The transmission coefficient T for the blunt saw-tooth of figure 6, as a function of β/α. (*a*) $c = \frac{1}{8}$, (*b*) $c = \frac{1}{4}$.

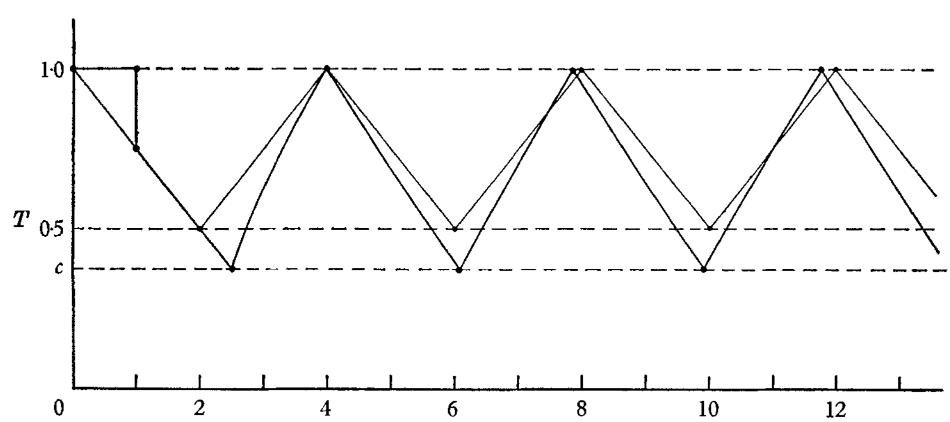
similar to the behaviour described in §4, that is to say T fluctuates between $\frac{1}{2}c$ and 1. The corresponding slope α of the incident rays may be called a critical slope. This is one for which the incident ray which passes through the point B in figure 6, after an integral number of reflexions, passes through the mid-point of the segment CD. However, now the critical slopes no longer occur at equal intervals of α^{-1}, but at intervals which become uniform as $\alpha \rightarrow \infty$.

For speed in calculating the transmission coefficient a computer program was written to calculate the ray paths and the corresponding coefficient T. The results are shown in figures 7 (a), (b) and (c) for $c = \frac{1}{8}$, $\frac{1}{4}$ and $\frac{1}{2}$.

It will be seen that, as β/α increases through 1, so T drops suddenly from unity to the value c. This is because the forward slopes of the wave-form are suddenly shielded from the incident rays, while the rear slopes reflect the rays backwards, not onwards. Only the rays falling on the two horizontal parts of the wave-form



(c)



(d)

FIGURE 7 (c), (d). The transmission coefficient T for a blunt saw-tooth. (c) $c = \frac{1}{2}$, (d) $c = \frac{3}{4}$. For comparison, figure 7 (d) shows also the transmission coefficient for a low, square-topped wave-form with $h/b = \frac{1}{4}\beta$, $b = \frac{1}{2}$ (thin curve).

are transmitted. As β/α increases further, T falls linearly from c to $\frac{1}{2}c$, after which it oscillates between $\frac{1}{2}c$ and 1, though not linearly in general.

As $c \to 0$ it is easy to see that for any given ray such that $\beta/\alpha > 1$ the number n of reflexions must increase without limit. In fact, if (x_i, z_i) are the Cartesian co-ordinates of the ith reflexion point, relative to axes with origin at the bottom of the 'unblunted' wave-form, we have:

$$z_i = \beta|x_i|, \quad (z_i - z_{i+1}) = \alpha(|x_i| + |x_{i+1}|). \tag{5.2}$$

So

$$\beta(|x_i| - |x_{i+1}|) = \alpha(|x_i| + |x_{i+1}|), \tag{5.3}$$

or

$$\frac{|x_i|}{|x_{i+1}|} = \frac{1-\alpha/\beta}{1+\alpha/\beta}. \tag{5.4}$$

Hence

$$\frac{|x_0|}{|x_n|} = \frac{(1-\alpha/\beta)^n}{(1+\alpha/\beta)^n} \doteqdot e^{2n\alpha/\beta}. \tag{5.5}$$

Since $|x_0| \doteqdot \frac{1}{2} - \frac{1}{4}c$ and $|x_n| \doteqdot \frac{1}{4}c$ we have,

$$(2-c)/c \doteqdot e^{2n\alpha/\beta}, \tag{5.6}$$

so

$$n \doteqdot (\beta/2\alpha)\log(2/c - 1). \tag{5.7}$$

Hence n is nearly proportional to $\log(1/c)$. As $c \to 0$, n increases without limit, but relatively slowly.†

Thus, if viscosity is ignored, then as $c \to 0$ the zig-zag curve in figure 7 becomes compressed towards the left like a concertina (figure 7*a*). Hence when $\beta/\alpha > 1$ the transmission coefficient T is indeterminate. We shall see in §8 how this indeterminacy is removed when viscosity is taken into account.

6. Reflexion from a sine-wave

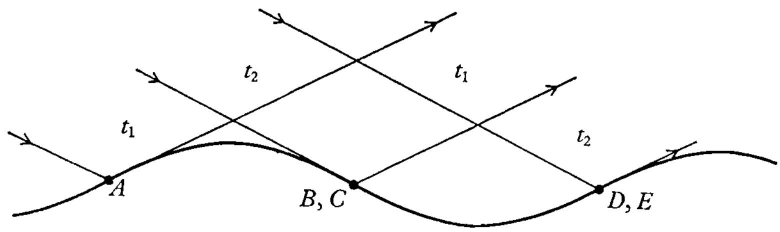
We are now ready to understand the reflexion of rays from a simple sine-wave (figure 8).

Let β denote the maximum slope of the sine-wave. When $\beta/\alpha < 1$ all rays are reflected onwards: $T = 1$. When $\beta/\alpha > 1$, a typical wavelength $ABCDE$ may be divided into four parts: a part AB near the crest, where the absolute slope is less than α; a similar part CD near the trough; and two parts BC and DE near the sides, where the absolute slope is greater than α. A typical ray striking AB will be reflected upwards and onwards. A ray striking DE will be reflected downwards, possibly more than once, until it reaches CD, when it will be reflected back upwards and either onwards or backwards according as the number of reflexions which it has undergone is even or odd.

There are two kinds of critical slopes α. When α is such that the ray through B (which is tangent to the wave-form) passes through the mid-point of CD after n reflexions, then *all* the rays incident between B and E are reflected backwards

† Carl Wunsch has pointed out to me a paper by Balazs (1961) from which it can be shown that the field of motion is periodic in $\log z$. See also Wunsch (1968).

(a)



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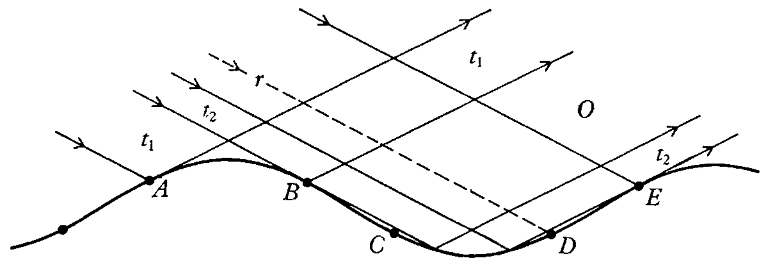

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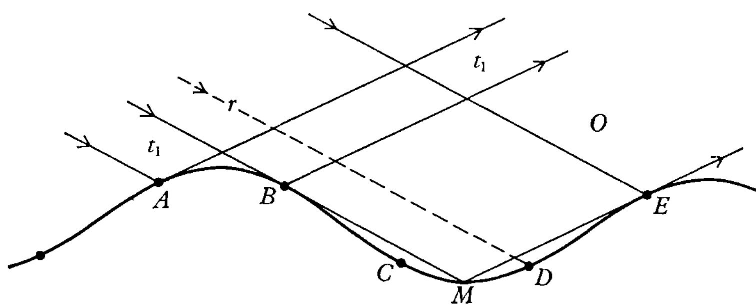

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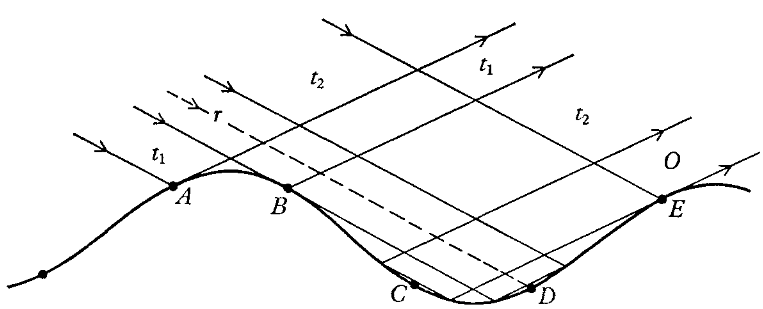


FIGURES 8(*a*)–(*d*). For legend see p. 244.

(as in figures 8 (*c*) and (*g*)). Thus the only transmitted rays are those falling on AB. Hence $T = \phi/\pi$ where $\beta \sin \phi = \alpha$, that is to say

$$T = (1/\pi) \sin^{-1} (\alpha/\beta). \tag{6.1}$$

Examples of this are illustrated in figures 8 (*c*) and (*g*). On the other hand, when the ray through B, after n reflexions, passes through one of the end-points of CD, then all the rays incident between B and E are ultimately reflected onwards. Hence

$$T = 1. \tag{6.2}$$

(Examples of this are illustrated in figures 8 (*a*) and (*e*).)

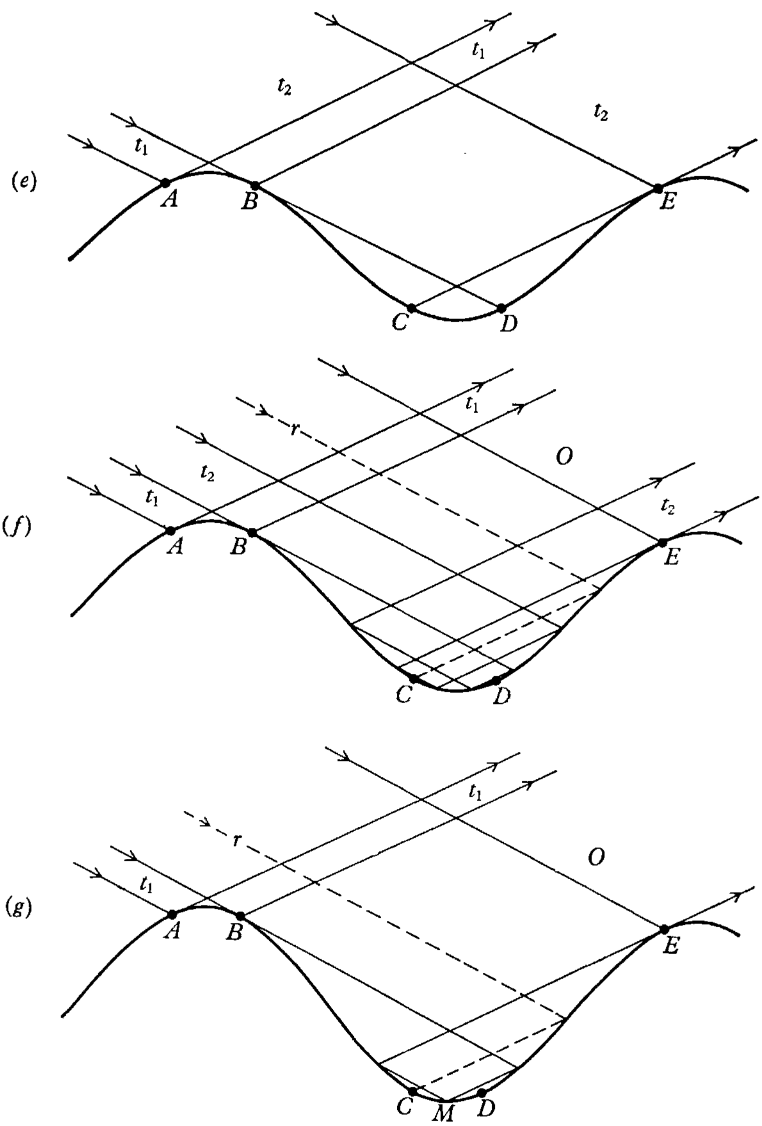


FIGURE 8(*e*)–(*g*). The reflexion of rays from a sine-wave.

In the intervals between these critical values the function T is piecewise monotonic, as can easily be seen from figure 8. Thus the transmission coefficient oscillates between the two bounding curves (6.1) and (6.2).

The calculated curve of T as a function of β/α is shown in figure 9. As might be expected, there is a close resemblance between this curve and figure 7(*b*).

The two bounding functions (6.1) and (6.2) are indicated in figure 9 by broken lines. We note that for large values of β/α (6.1) becomes

$$T \doteqdot \frac{1}{\pi}\frac{\alpha}{\beta}, \tag{6.3}$$

which tends to zero as $\beta/\alpha \to \infty$; this is in contrast to the lower bounding curves of figure 7, which correspond to $T = \frac{1}{2}c$. Evidently the difference is due to the fact that for a sine-wave the length of that part of the curve for which the slope is less than α diminishes to zero as $\alpha \to 0$; but for the flat-topped curve there is always such a segment of length $\frac{1}{2}c$.

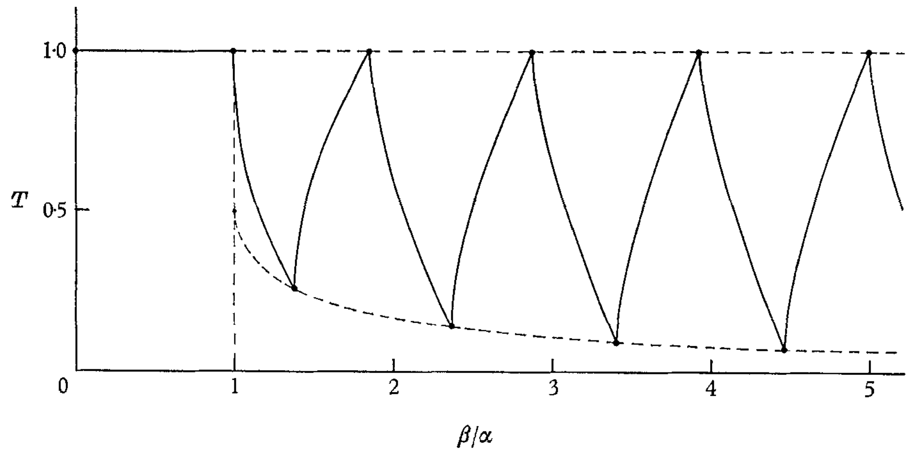


FIGURE 9. The transmission coefficient T of a simple sine-wave as a function of the ratio of the maximum slope β to the slope α of the incident rays.

7. A modulated sine-wave

A moment's consideration will show that the reflecting properties of an irregular wave pattern such as in figure 10(a) will be highly complicated; the chief complexity arising from the possibility that rays reflected from the trough of one wave may meet the sides of an adjacent trough.

If, however, we consider a wave train of slowly varying amplitude and phase such as in figure 10(b), we may assume that the reflexion of a given ray from the sides of more than one trough is negligible provided that the slope of the wave envelope (shown by a broken line) is smaller than the slope of the incident rays. Let us suppose this to be so, and let us consider for the sake of illustration the case when the wave-form has a narrow spectrum, with a Rayleigh distribution of wave amplitude. Such a wave-form is characteristic of many physical phenomena, including surface waves on the ocean (see Longuet-Higgins 1952; Cartwright & Longuet-Higgins 1956). Then, the wavelength being almost constant, the steepest slope β of each individual wave trough is proportional to the amplitude of the individual trough, and so has a Rayleigh distribution also. The probability density of β is therefore given by

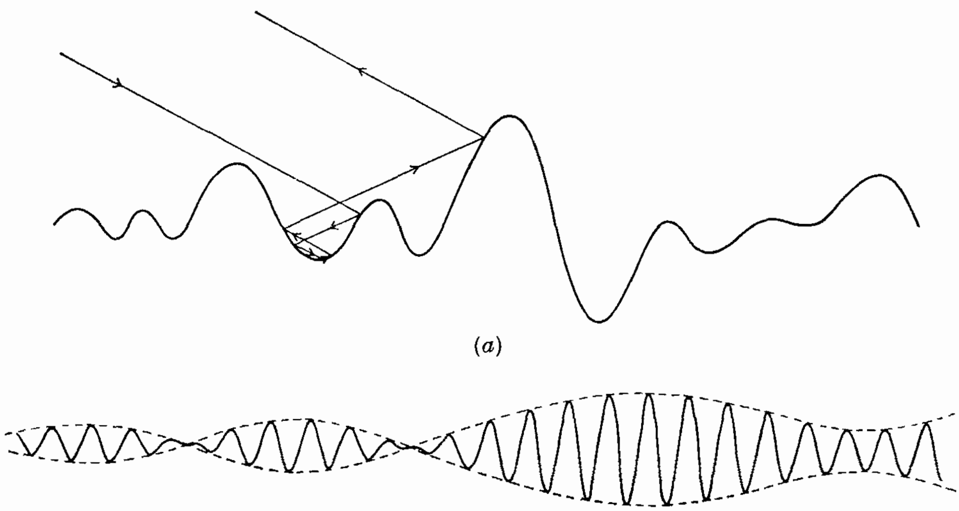
$$p(\beta) = dP/d\beta, \tag{7.1}$$

where
$$P(\beta) = -\exp(-\beta^2/\bar{\beta}^2)$$

and $\bar{\beta}$ denotes the r.m.s. value of β.

Let us find the mean value $\overline{T}$ of the transmission coefficient T when β is distributed according to the law (7.1). This is given simply by

$$\overline{T} = \int_{\beta=0}^{\infty} T(\beta)\,dP, \tag{7.2}$$



(a)

(b)

FIGURE 10. (a) An irregular wave surface; (b) a gradually modulated sine-wave.

where $T(\beta)$ denotes the function of figure 9, and P is given by (7.1). Now $T(\beta)$ is somewhat complicated, but as we have seen it oscillates between two fairly simple limiting curves. We shall approximate $T(\beta)$ by the mean value of these two curves; that is to say we take

$$T(\beta) = \begin{cases} 1 & (\beta/\alpha) \leqslant 1), \\ \tfrac{1}{2}[1+(1/\pi)\sin^{-1}(\alpha/\beta)] & (\beta/\alpha > 1). \end{cases} \tag{7.3}$$

Substituting (7.3) into (7.2) we obtain

$$\overline{T} = \int_{\beta=0}^{\alpha} dP + \int_{\beta=\alpha}^{\infty} \tfrac{1}{2}[1+(1/\pi)\sin^{-1}(\alpha/\beta)]\,dP. \tag{7.4}$$

On integrating the second term by parts we have

$$\overline{T} = [P(\alpha)-P(0)] - \tfrac{3}{4}P(\alpha) - \frac{1}{2\pi}\int_{\alpha}^{\infty} \frac{P(\beta)}{\sqrt{(1-\alpha^2/\beta^2)}}\,d(\alpha/\beta). \tag{7.5}$$

Writing
$$\alpha/\beta = 1/\xi, \quad \alpha/\overline{\beta} = \mu, \tag{7.6}$$

and substituting for $P(\beta)$ from (7.1), we find

$$\overline{T} = (1-\tfrac{1}{4}e^{-\mu^2}) - \frac{1}{2\pi}\int_{1}^{\infty} \frac{e^{-\mu^2\xi^2}}{\xi\sqrt{(\xi^2-1)}}\,d\xi. \tag{7.7}$$

On writing $\xi^2 = 1+\zeta$ the integral is transformed as follows:

$$\int_1^\infty \frac{e^{-\mu^2\xi^2}}{\xi\sqrt{(\xi^2-1)}}\,d\xi = \tfrac{1}{2}e^{-\mu^2}\int_0^\infty \frac{e^{-\mu^2\zeta}}{(1+\zeta)\sqrt{\zeta}}\,d\zeta = \tfrac{1}{2}\pi\,\mathrm{erfc}\,(\mu), \tag{7.8}$$

where
$$\mathrm{erfc}\,(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}\,dt = 1-\mathrm{erf}\,(x) \tag{7.9}$$

(see Erdélyi *et al.* 1954, p. 136). Thus altogether we find

$$\bar{T} = (1-\tfrac{1}{4}e^{-\mu^2})-\tfrac{1}{4}(1-\mathrm{erf}\,(\mu)), \tag{7.10}$$

that is
$$\bar{T} = \tfrac{1}{4}[3-\exp\,(-\alpha^2/\bar{\beta}^2)+\mathrm{erf}\,(\alpha/\bar{\beta})]. \tag{7.11}$$

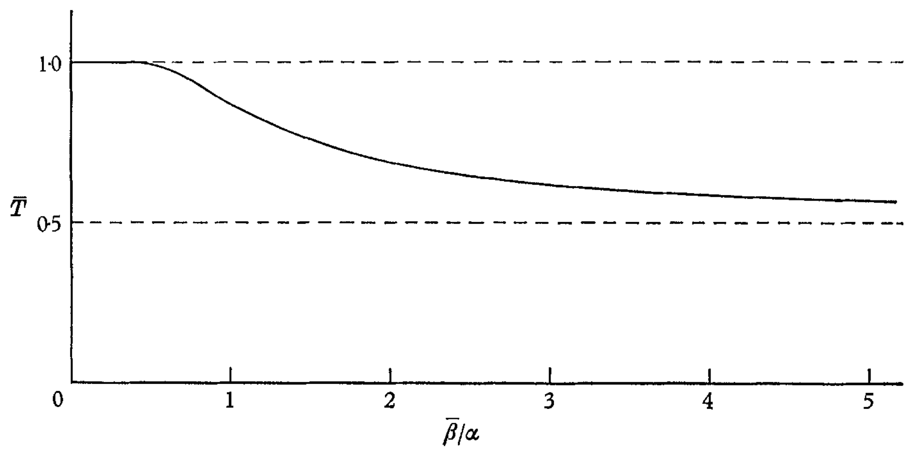


FIGURE 11. The mean transmission coefficient $\bar{T}$ for a rough surface having the form of a gradually modulated sine-wave with a Gaussian distribution of surface slope. $\bar{\beta}$ denotes the r.m.s. value of β, the steepest slope of an individual wave.

The function (7.11) is shown in figure 11. It will be seen that $\bar{T}$ is practically equal to unity when $\bar{\beta}/\alpha < 0{\cdot}5$, that is to say when the r.m.s. 'slope amplitude' is less than half the slope of the incident rays, but that for larger values of $\bar{\beta}/\alpha$ the transmission coefficient falls off rather rapidly.

One must bear in mind that the r.m.s. value of the local slope s is not equal to $\bar{\beta}$ but to $\bar{\beta}/\sqrt{2}$. It is possible that (7.11) may be valid for non-Gaussian reflecting surfaces, provided that $\bar{\beta}$ is replaced by $\sqrt{2}\,\bar{s}$. For some kinds of surface, however, $\bar{s}$ may not even exist. A more plausible generalization of (7.11) is to express $\bar{\beta}$ in terms of the median value s_1 of $|s|$. For the Gaussian surface just described, this is given by

$$\mathrm{erf}\,(s_1/\bar{\beta}) = 0{\cdot}5, \tag{7.12}$$

and hence
$$s_1/\bar{\beta} = 0{\cdot}4769\,\ldots. \tag{7.13}$$

This gives
$$\bar{T} = \tfrac{1}{4}(3-e^{-(C\alpha/s_1)^2}+\mathrm{erf}\,(C\alpha/s_1)], \tag{7.14}$$

where s_1 denotes the median of the distribution of absolute slope and $C \doteqdot 0{\cdot}477$. It would be most interesting to compare (7.14) with experimental determinations of the transmission coefficient for various kinds of irregular reflecting surfaces.

8. Viscous effects

Let us consider how the situation shown in figure 1 will be influenced by viscous forces.

In the absence of viscosity the particle motion close to the boundary is not zero, but has an oscillating component parallel to the boundary. However, the viscous boundary condition implies that both normal and tangential components of velocity vanish. For sufficiently small viscosity (i.e. high Reynolds number) the boundary conditions can be satisfied by adding to the non-viscous solution a thin boundary layer, whose thickness is of order

$$\delta = (\nu/\sigma)^{\frac{1}{2}}, \tag{8.1}$$

where ν denotes the kinematic viscosity and σ denotes the radian frequency.

However, because the velocity just outside the boundary layer is a harmonic function of distance parallel to the boundary, there is a non-uniform flux within the layer. Hence, by continuity there is a small component of velocity normal to the boundary just outside the layer. This component of velocity is of order δ/λ times the tangential component, where λ denotes the tangential wavelength. The normal velocity in turn induces a small change in the amplitude of the reflected wave and also a small phase shift, both of order δ/λ.

The order of magnitude of the energy loss is easily found from the consideration that virtually all the energy dissipation takes place within the boundary layer. One may assume that in a single wave cycle a fraction of order unity of the energy in this layer is destroyed. But in one wave cycle a fraction of order unity of the energy in a typical wavelength of the incident or reflected wave is also transmitted, since the group velocity is of the same order as the phase velocity. Therefore, the proportion of energy lost is of order δ/λ, where λ denotes the smaller of the wavelengths of the incident and reflected waves.

When either the incident or the reflected ray is nearly parallel to the boundary, it can be seen that the corresponding wavelength λ of that ray is necessarily much reduced; hence the energy loss is high.

In addition, when δ/λ is $O(1)$ in either the incident or reflected rays, the shearing velocities in the region beyond the boundary layers will become so large as to produce considerable energy losses there also; in fact, the boundary-layer approximation will no longer apply.

Generally, we see that the energy loss will be slight provided that δ/λ is small and the difference between the slope of the boundary and the slope of the incident or reflected rays is not of order δ/λ or less. Nevertheless, any ray that has been reflected a number of times of order λ/δ will lose a substantial amount of energy. Such a ray reflected an infinite number of times will be eventually extinguished.

Hence in § 5 the correct limiting form of T as $c \to 0$ is indeed the form shown in figure 3, rather than the indeterminate form of figure 7 as $c \to 0$. Further, we can approximate the value of δ at which the non-viscous ray theory of figure 6 becomes inapplicable. This will be after a number of reflexions n such that

$$n\delta/\beta\lambda = O(1).$$

Because of viscous losses, the values of T calculated in §§ 2–7 on the non-viscous theory must be considered as upper bounds. The greater the number n of reflexions, the more the viscous transmission coefficient will diverge from the non-viscous coefficient.

9. Discussion

From the previous section it can be seen that viscosity limits the accuracy of the ray theory at least to those situations where the relevant length scales are all large compared to δ. However, the scale of the surface roughness may be quite small compared to the *wavelength* λ of the incident or reflected waves, and the complicated patterns of reflexion described in §§ 2–7 will still be found, provided only that roughness scale is large compared to δ.

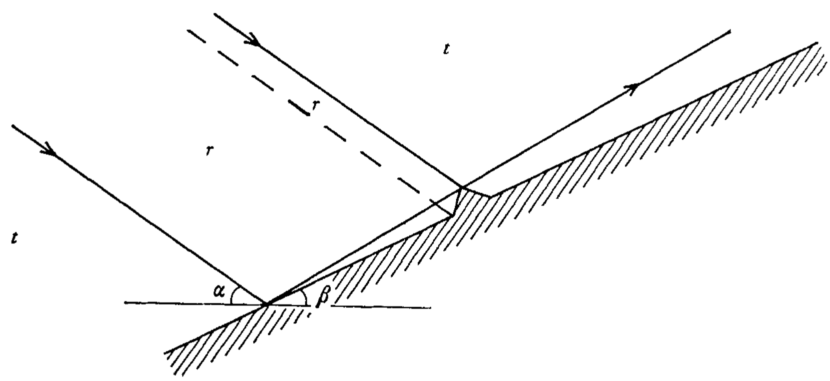


FIGURE 12. The effect of a projection on the reflexion of waves from a sloping bottom inclined at nearly the critical angle.

In other words, complicated reflexions may be expected from all roughness scales which are large compared to δ, no matter how great the wavelength of the incident radiation. Only those roughness scales that are small compared to δ, being 'blanketed' by the boundary layer, will be too small to affect the reflexion from the surface.

We may remark that, for waves of period 12 h in water of kinematic viscosity $\nu = 0{\cdot}013\,\mathrm{cm^2/sec}$, we have $\sigma = 1{\cdot}3 \times 10^{-4}\,\mathrm{rad/sec}$, and so

$$\delta = (\nu/\sigma)^{\frac{1}{2}} = 10\,\mathrm{cm}.$$

Thus even for waves of tidal period the reflexion may be complicated by irregularities on the ocean floor of the order of 1 m or more.

This conclusion must be subject to the consideration that the probable increase in wave amplitude near the bottom, accompanied by strong shear, may cause the motion to become locally unstable and hence produce turbulent mixing. In this way the basic stratification of the lower layer could be destroyed, leading to the onwards reflexion of the incident rays before they had reached the bottom.

However, this effect would not be felt by a wave motion that depended solely on the rotation, such as inertial wave motion in a homogeneous fluid.

It is not difficult in principle to extend the results of the present paper to rough surfaces whose mean inclination to the horizon is different from zero, or to systems in which the incident and reflected rays make different angles with the vertical direction, or to situations which are three-dimensional. These will be left as recreations for the reader.

Sandstrom (1966) has shown experimentally that, when an incoming wave impinges on a smooth inclined surface that has almost the same inclination as one of the ray paths, then an intense beam is transmitted (as would indeed be expected from figure 1 (*b*)). He suggested that near the edge of the continental shelf, for example, barotropic tidal motions might be efficiently converted into baroclinic tides. However, a simple extension of the present analysis to a rough but inclined surface shows that, when the mean inclination of the surface is nearly that of a characteristic ray, only a slight roughness will result in the absorption or scattering of a large part of the transmitted beam (see figure 12). In practice, therefore, Sandstrom's conclusions may require substantial modification.

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REFERENCES

BALAZS, N. L. 1961 On the solution of the wave equation with moving boundaries. *J. Math. Anal. Applic.* **3**, 472–84.

CARTWRIGHT, D. E. & LONGUET-HIGGINS, M. S. 1956 On the distribution of the maxima of a random function. *Proc. Roy. Soc.* A **237**, 212–32.

ECKART, C. H. 1960 *Hydrodynamics of Oceans and Atmospheres*. Oxford: Pergamon.

ERDÉLYI, A., OBERHETTINGER, F. & MAGNUS, J. A. 1954 *Tables of Integral Transforms*, vol. 1. New York: McGraw-Hill.

FOFONOFF, N. P. 1967 Internal tide generation and propagation from continental shelves. I. Ray Theory. *Proceedings IAPO Symposium on Internal Waves, General Assembly of IUGG, Berne, Switzerland.* (Abstract only.)

GREENSPAN, H. 1965 On the general theory of contained rotating motions. *J. Fluid Mech.* **22**, 449–62.

LONGUET-HIGGINS, M. S. 1952 On the statistical distribution of the heights of sea waves. *J. Mar. Res.* **11**, 245–66.

MOWBRAY, D. F. & RARITY, B. S. H. 1967 A theoretical and experimental investigation of the phase configuration of internal waves of small amplitude in a density stratified liquid. *J. Fluid Mech.* **28**, 1–16.

SANDSTROM, H. 1966 The importance of topography in generation and propagation of internal waves. Ph.D. Dissertation, University of California, San Diego.

WUNSCH, C. 1968 On the propagation of internal waves up a slope. *Deep Sea Res.* **15**, 251–8.